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Saturation of Positive Operators*

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1. HISTORICAL INTRODUCTION

The theory of saturation of nontrigonometric operators originated with the results about Bernstein polynomials $B_n(f, x)$ by DeLeeuw [6], by one of the present authors [13], and by Bajanski and Bojanić [2]. In particular, Lorentz [13] proved that $| B_n(f, x) - f(x)| \leqslant [x(1 - x)/2n] M$, $0 \leqslant x \leqslant 1$, if and only if the function f has a derivative that belongs to the class $\mathrm{Lip}_M 1$. Bajanski and Bojanić proved that if $B_n(f, x) - f(x) = o(n^{-1})$ at each point $0 \leqslant x \leqslant 1$, then f is linear.

The method of the paper [13] has been further developed by DeLuca [7] on one hand, and by Ikeno, Suzuki and Watanabe [8, 21–23] on the other. They showed that the proof of [13] can be formulated in a more general form, which allows one to apply it with success to many concrete problems.

The method of Bajanski and Bojanić found further development in the papers of Amel'kovic [1] and Mühlbach [15]. They have noticed that asymptotic relations (of the type appearing in the theorem of Voronovskaja, see [12, p. 22]) are of great help in this problem.

Recently, the method of semigroups of operators and of infinitesimal generators has been applied to the saturation problems by Schnabl [18], Micchelli [14] and Karlin and Ziegler [11]. (Butzer was the first to discover the usefulness of this method in trigonometric saturation problems; see for example, Butzer and Berens [3]).

In the present paper we shall further develop the second method. Since there does not exist a comparative study of the merits of the three methods described above, let us stress its advantages: it leads to simple proofs (Section 4), allows local assumptions to be treated (compare $o(1)$ in

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Theorem 4.3, which is not uniform), and it can be applied to a wide variety of known special approximation operators (sections 5, 6).

Basic for us will be the existence of an appropriate asymptotic formula and the positivity of the approximating operator. An important tool will be the theory of Čebyšev systems, developed by Karlin and Studden [10]. As a byproduct of our results, we shall see that a sequence of positive linear operators cannot satisfy an asymptotic relation of the form (4.1) with $k > 2$.

2. Differential operators, Čebyšev systems and convexity

Let $w_0, \ldots, w_k$ be continuous, strictly positive functions on (a, b), such that $w_j \in C^{k-j}$, $j = 0, \ldots, k$. We define the operators

$$D_j f(x) = [(1/w_j(x)) f(x)]', \qquad j = 0, \ldots, k-1, \tag{2.1}$$

$$Df(x) = [1/w_k(x)]\, D_{k-1} \cdots D_0 f(x), \tag{2.2}$$

and

$$\Delta f(x) = [1/w_{k-1}(x)]\, D_{k-2} \cdots D_0 f(x). \tag{2.3}$$

With the operator D we associate the functions (where c is a fixed point, $a < c < b$)

$$\begin{aligned} u_0(x) &= w_0(x), \\ u_1(x) &= w_0(x) \int_c^x w_1(t_1)\, dt_1, \\ &\cdots \\ u_k(x) &= w_0(x) \int_c^x w_1(t_1)\, dt_1 \int_c^{t_1} w_2(t_2)\, dt_2 \cdots \int_c^{t_{k-1}} w_k(t_k)\, dt_k . \end{aligned} \tag{2.4}$$

The functions $u_0, \ldots, u_{k-1}$ form a fundamental set of solutions of the differential equation $Du = 0$; the function u_k satisfies $Du_k \equiv 1$.

According to Karlin and Studden [10, Chap. 11] the functions (2.4) form an extended complete Čebyšev system. This means that they belong to $C^k(a, b)$ and that each set $u_0, \ldots, u_j$, $0 \leqslant j \leqslant k$, is a Čebyšev system on each closed subinterval of (a, b). In particular, the determinants

$$U\binom{u_0, \ldots, u_j}{x_0, \ldots, x_j} = \begin{vmatrix} u_0(x_0) & u_0(x_1) & \cdots & u_0(x_j) \\ u_1(x_0) & u_1(x_1) & \cdots & u_1(x_j) \\ & \cdots & & \\ u_j(x_0) & u_j(x_1) & \cdots & u_j(x_j) \end{vmatrix} > 0, \tag{2.5}$$

whenever $a < x_0 < \cdots < x_j < b$, $0 \leqslant j \leqslant k$.

With the system (2.4) we sometimes associate the *reduced system* $v_0, \ldots, v_{k-1}$, defined by

$$v_j(x) = [u_{j+1}(x)/u_0(x)]', \qquad j = 0, \ldots, k-1. \tag{2.6}$$

The functions v_j belong to C^{k-1}, and are given by formulas similar to (2.4), involving the functions $w_1, \ldots, w_k$; they too form an extended complete Čebyšev system. Moreover [10, p. 383],

$$U\begin{pmatrix} u_0, \ldots, u_k \\ x_0, \ldots, x_k \end{pmatrix} = \prod_{j=0}^{k} u_0(x_j) \int_{x_0}^{x_1} \int_{x_1}^{x_2} \cdots \int_{x_{k-1}}^{x_k} U\begin{pmatrix} v_0, \ldots, v_{k-1} \\ t_0, \ldots, t_{k-1} \end{pmatrix} dt_0\, dt_1 \cdots dt_{k-1}. \tag{2.7}$$

A function ϕ of the form $\phi = \sum_{j=0}^{k-1} a_j u_j$ will be called *linear*. A function f on (a, b) is *convex* with respect to $u_0, \ldots, u_{k-1}$ if $k \geqslant 2$, and if

$$U\begin{pmatrix} u_0, \ldots, u_{k-1}, f \\ x_0, \ldots, x_{k-1}, x_k \end{pmatrix} \geqslant 0, \qquad a \leqslant x_0 < \cdots < x_k \leqslant b. \tag{2.8}$$

For $i = 2, 3, \ldots, k$, the function u_i is convex with respect to $u_0, \ldots, u_{i-1}$. An important theorem of Karlin and Studden [10] characterizes convex functions f. (A simple proof of this theorem, due to Mühlbach [17], will appear in this Journal.) The relevant properties are: $f \in C^{k-2}(a, b)$ and f has right and left derivatives $f_R^{(k-1)}(x)$, $f_L^{(k-1)}(x)$ for $a < x < b$, which are, respectively, right and left continuous. Moreover,

$$\Delta_R f(x) = [1/w_{k-1}(x)]\, D_{k-2}^R\, D_{k-3} \cdots D_0 f(x) \tag{2.9}$$

is right continuous and increasing on (a, b).

For any function $f(x)$ on (a, b) and any set of points

$$a \leqslant x_0 < \cdots < x_{k-1} \leqslant b,$$

there exists a unique interpolating function $\phi = \sum_0^{k-1} a_j u_j$, which coincides with f for $x = x_j$, $j = 0, \ldots, k-1$. The following two lemmas will be used in Section 4.

LEMMA 2.1. *A continuous function f is convex on (a, b) with respect to u_0, u_1 if and only if for every $a < x_1 < x_2 < b$ the linear function $\phi = a_0 u_0 + a_1 u_1$ interpolating f at x_1, x_2 satisfies $\phi(x) \leqslant f(x)$ for all $x_1 \leqslant x \leqslant x_2$.*

LEMMA 2.2. *Let f be convex with respect to u_0, u_1 and let ϕ be the linear function such that $\phi(x_0) = f(x_0)$ and $\phi'(x_0) = f_R'(x_0)$. Then $\phi(x) \leqslant f(x)$ for all $a \leqslant x \leqslant b$.*

The simple proofs are left to the reader. Compare also [9, Theorem 2.2, p. 282].

3. Lipschitz Conditions

We shall say that a function h, continuous on (a, b), satisfies the Lipschitz condition or belongs to the class $\mathrm{Lip}_M{}^{+}1$ with respect to $u_0, ..., u_k$, with $M \geqslant 0$, if h has a right continuous derivative $h_R^{(k-1)}$ and a left continuous $h_L^{(k-1)}$ and if

$$\Delta h(x_1) - \Delta h(x_0) \geqslant -M \int_{x_0}^{x_1} w_k(t)\, dt, \qquad a < x_0 < x_1 < b, \tag{3.1}$$

where for each of the values of Δh one can substitute $\Delta_R h$ and $\Delta_L h$. Similarly, h belongs to the class $\mathrm{Lip}_M{}^{-}1$, if

$$\Delta h(x_1) - \Delta h(x_0) \leqslant M \int_{x_0}^{x_1} w_k(t)\, dt. \tag{3.2}$$

Finally, h belongs to the class $\mathrm{Lip}_M 1$ with respect to the-system $u_0, ..., u_k$ if $h^{(k-1)}$ is absolutely continuous, and if almost everywhere

$$| Dh(x)| \leqslant M. \tag{3.3}$$

Lemma 3.1. *A continuous function f on (a, b) belongs to the class* $\mathrm{Lip}_M{}^{+}1$ *with respect to $u_0, ..., u_k$ if and only if $Mu_k + f$ is convex with respect to $u_0, ..., u_{k-1}$.*

Proof. If $Mu_k + f$ is convex, the required smoothness of f follows from the theorem of Karlin and Studden mentioned in Section 2.

The condition that $Mu_k + f$ is convex is equivalent to the validity of the inequality

$$U\begin{pmatrix} u_0, ..., u_{k-1}, f \\ x_0, ..., x_{k-1}, x_k \end{pmatrix} \geqslant -MU\begin{pmatrix} u_0, ..., u_{k-1}, u_k \\ x_0, ..., x_{k-1}, x_k \end{pmatrix}, \tag{3.4}$$

for all possible points $a < x_0 < \cdots < x_k < b$. To each of the two determinants U in this inequality, we apply the formula (2.7); for the first of them we replace u_k by f and v_{k-1} by $D_0 f$. Then, using the positivity of w_0 and the fact that the intervals $(x_0, x_1), ..., (x_{k-1}, x_k)$ are disjoint, we see that (3.4) holds for all selections of the points $a < x_0 < \cdots < x_k < b$, if and only if

$$U\begin{pmatrix} v_0, \ldots, v_{k-2}, D_0 f \\ t_0, \ldots, t_{k-2}, t_{k-1} \end{pmatrix} \geqslant -MU\begin{pmatrix} v_0, \ldots, v_{k-1} \\ t_0, \ldots, t_{k-1} \end{pmatrix} \tag{3.5}$$

holds for all selections of $a < t_0 < \cdots < t_{k-1} < b$.

We can repeat this process, obtaining reduced systems [see (2.6)] of higher order. After $k - 1$ steps we reduce (3.4) to

$$U\begin{pmatrix} w_{k-1}, & w_{k-1} \Delta f \\ x_0, & x_1 \end{pmatrix} \geqslant -MU\begin{pmatrix} w_{k-1}, & w_{k-1} \int w_k \\ x_0, & x_1 \end{pmatrix}.$$

(We use here the fact that a continuous function is the integral of its right derivative if the latter exists everywhere and is integrable). But this is equivalent to

$$\Delta f(x_1) - \Delta f(x_0) \geqslant -M \int_{x_0}^{x_1} w_k(t)\, dt.$$

THEOREM 3.2. *The following are equivalent for $f \in C(a, b)$:*

$$f \in \mathrm{Lip}_M{}^{+}1 \cap \mathrm{Lip}_M{}^{-}1 \quad \textit{with respect to } u_0, \ldots, u_k. \tag{3.6}$$

$$\Delta f \textit{ is absolutely continuous on } (a, b) \textit{ and} \tag{3.7}$$

$$| \Delta f(x_1) - \Delta f(x_0)| \leqslant M \int_{x_0}^{x_1} w_k(t)\, dt.$$

$$f \in \mathrm{Lip}_M 1 \quad \textit{with respect to } u_0, \ldots, u_k. \tag{3.8}$$

Proof. The inequality (3.7) is equivalent to (3.6) by definition. If in this inequality we make x_0 and x_1 approach, from left and right, some point x of (a, b), we see that $\Delta_L f(x) = \Delta_R f(x)$ at this point. Thus (3.7) is equivalent to the absolute continuity of $f^{(k-1)}$ and to $| Df(x)| \leqslant M$ a.e.

4. THE SATURATION CLASSES

We assume that L_n, $n = 1, 2, \ldots$, are linear positive operators that map $C[a, b]$ into itself. In addition, we assume that L_n satisfy the following asymptotic fomula:

$$\lim_{n\to\infty} \lambda_n[L_n f(x) - f(x)] = \rho(x)\, Df(x), \qquad a < x < b, \tag{4.1}$$

where $\lambda_n > 0$ converges to $+\infty$ with n, $\rho(x) \geqslant 0$ on $[a, b]$, and $\rho(x) > 0$ on

(a, b), and D is a differential operator of type (2.2). We assume that (4.1) holds whenever f belongs to the class C^k (where k is the degree of D) in a neighborhood of the point x.

In the following lemma we write $r_n(f) = L_n(f) - f$.

LEMMA 4.1. *If*

$$\lambda_n r_n(f, x) \geqslant -M\rho(x) + o(1), \qquad a < x < b, \tag{4.2}$$

then the function $Mu_k + f$ is convex with respect to u_0, u_1.

Proof. Assume that $Mu_k + f$ is not convex. Then, since the set of all convex functions is closed in the space of continuous functions, the function $g = f + (M + \epsilon)\, u_k$ is not convex for some $\epsilon > 0$.

According to Lemma 2.1, this implies the following. There exist points $a < x_1 < y < x_2 < b$ such that for the function ϕ interpolating g at x_1, x_2, $\phi(y) < g(y)$. We consider the function

$$[g(x) - \phi(x)]/w_0(x). \tag{4.3}$$

It is continuous on $I = [x_1, x_2]$, vanishes at the endpoints x_1, x_2, and has value >0 at y. Let m be its maximum on I; suppose it is achieved at z with $x_1 < z < x_2$. The linear function $\phi_1(x) = \phi(x) + mw_0(x)$ coincides with g at z and satisfies $g(x) \leqslant \phi_1(x)$ for $x \in I$.

We can extend g from I to a continuous function G on $[a, b]$, for which

$$\begin{aligned} &G(z) = \phi_1(z), \\ &G(x) \leqslant \phi_1(x), \qquad a \leqslant x \leqslant b, \\ &G(x) = g(x), \qquad x \in I \end{aligned} \tag{4.4}$$

Then, applying (4.1) to the function $g - G$ and to u_k (for which $Du_k \equiv 1$), we obtain

$$\begin{aligned} \lambda_n r_n(f, z) &= \lambda_n r_n(f + (M + \epsilon)\, u_k, z) - (M + \epsilon)\, \lambda_n r_n(u_k, z) \\ &= \lambda_n r_n(G, z) - (M + \epsilon)\, \rho(z) + o(1). \end{aligned}$$

Since L_n is positive, it follows by (4.4) and (4.1) that

$$\begin{aligned} \lambda_n r_n(f, z) &= \lambda_n[L_n(G, z) - G(z)] - (M + \epsilon)\, \rho(z) + o(1) \\ &\leqslant \lambda_n[L_n(\phi_1, z) - \phi_1(z)] - (M + \epsilon)\, \rho(z) + o(1) \\ &\leqslant -(M + \epsilon)\, \rho(z) + o(1). \end{aligned}$$

But this contradicts (4.2). This completes the proof of Lemma 4.1.

Similarly, if

$$\lambda_n r_n(f, x) \leqslant M\rho(x) + o(1),$$

then the function $Mu_k - f$ is convex.

As a corollary we have

THEOREM 4.2. *A sequence of positive linear operators L_n cannot satisfy an asumptotic formula of the form* (4.1) *with $k > 2$.*

Proof. Assume the contrary. Then (4.1) implies that (4.2) is satisfied for $M = 0$ and $f = -u_2$. But then $-u_2$ is convex with respect to u_0, u_1, a contradiction.

From now on we shall assume $k = 2$ in (4.1). The following theorem identifies the saturation class of the operators L_n.

THEOREM 4.3. *A function $f \in C[a, b]$ satisfies*

$$\lambda_n \mid L_n f(x) - f(x)| \leqslant M\rho(x) + o(1), \qquad n \to \infty, \qquad a < x < b, \quad (4.5)$$

(*where $o(1)$ converges to zero for $n \to \infty$, but not necessarily uniformly*), *if and only if $f \in \mathrm{Lip}_M\, 1$ with respect to u_1, u_1, u_2.*

The theorem follows from the results of Section 3, Lemma 4.1 and the following lemma.

LEMMA 4.4. *If $Mu_2 + f$ is convex with respect to u_0, u_1, then* (4.2) *is satisfied.*

Proof. Let $a < x_0 < b$ be fixed. By Lemma 2.2, there exists a linear function ϕ with the properties

$$\begin{aligned} g(x_0) &= \phi(x_0), \\ g(x) &\geqslant \phi(x), \qquad a \leqslant x \leqslant b, \end{aligned} \quad (4.6)$$

where $g = Mu_2 + f$. Then

$$\lambda_n r_n(g, x_0) = \lambda_n[L_n(g, x_0) - \phi(x_0)] \geqslant \lambda_n[L_n(\phi, x_0) - \phi(x_0)] = o(1),$$

by (4.1). Hence

$$\lambda_n r_n(f, x_0) \geqslant -\lambda_n M r_n(u_2, x_0) + o(1) = -M\rho(x_0) + 0(1),$$

as required.

Remark 1. From the case $M = 0$ of Theorem 4.3 we obtain: If $\lambda_n r_n(f, x) = o(1)$ for $a < x < b$, then f is linear. This is a result of Bajanski–Bojanić type.

Remark 2. Sometimes it is possible to prove a slightly more satisfying statement, namely, that for $f \in C[a, b]$ one has

$$\lambda_n \mid L_n(f, x) - f(x)| \leqslant M\rho(x), \qquad a < x < b, \qquad n = 1, 2, \ldots, \tag{4.7}$$

if and only if $f \in \mathrm{Lip}_M 1$. This is true if L_n reproduces the functions u_0, u_1, and (4.7) holds for $f = u_2$.

Only the sufficiency of the condition is to be shown; the proof is the same as that of Lemma 4.4.

5. Approximation of Functions of Two Variables

Let $G_0 \subset R^2$ be an open domain with compact closure G. We consider a sequence of positive linear operators L_n that map $C(G)$ into itself and satisfy an asymptotic formula

$$\lim_{n \to \infty} \lambda_n [L_n(f; x, y) - f(x, y)] = \rho(x, y)\, Df(x, y), \qquad (x, y) \in G_0, \tag{5.1}$$

where $\lambda_n \to +\infty$, $\rho(x, y) > 0$, and D is an elliptic operator

$$Df = a(x, y)(\partial^2 f/\partial x^2) + 2b(x, y)(\partial^2 f/\partial x\, \partial y) + c(x, y)(\partial^2 f/\partial y^2) \tag{5.2}$$

with continuously differentiable functions a, b, c in G that satisfy $b^2 - ac < 0$ in G_0.

We shall need the following definitions and facts about operators (5.2) (see, for example, [5]). There exists a solution $U \in C^2(G_0)$ of the equation $DU \equiv 1$. A solution ϕ of $D\phi = 0$ will be called *harmonic*. For each disc $\gamma \subset G_0$, and for arbitrary continuous boundary values on the boundary of γ, the Dirichlet problem for γ is solvable with a harmonic ϕ (since the assumptions on D assure it is uniformly elliptic on the disc γ). A function $v \in C^2(G_0) \cap C(G)$ will be called *subharmonic*, if for each $\gamma \subset G_0$ one has $v \leqslant \phi$ on the disc bounded by γ, where ϕ is the solution of the Dirichlet problem with boundary values v on γ. The subharmonic functions are characterized by the inequality $Dv(x, y) \geqslant 0$ on G_0.

After these preparations, we can solve the saturation problem, at least for smooth functions f:

THEOREM 5.1. *The class of all functions $f \in C^2(G_0) \cap C(G)$ that satisfy*

$$\lambda_n \mid L_n(f; x, y) - f(x, y)| \leqslant M\rho(x, y) + o(1), \qquad (x, y) \in G_0 \tag{5.3}$$

is identical with the class of all functions for which both $MU + f$ and $MU - f$ are subharmonic.

The proof of the necessity of the conditions is the same as that of Lemma 4.1. If they are satisfied, then $D(MU \pm f) \geqslant 0$. This implies that $|Df| \leqslant MD(U) = M$. Then (5.3) follows from (5.1).

6. Application and Examples

For sequences of positive linear operators operating on functions of one variable, the saturation theorem of Section 4 permits the derivation of many known results as well as a wide variety of new ones. Many positive linear operators satisfy an asymptotic relation of the form

$$\lim_{n\to\infty} \lambda_n[L_n(f, x) - f(x)] = \alpha(x)f'(x) + \beta(x)f''(x), \; a < x < b, \qquad (6.1)$$

where $\alpha(x) \geqslant 0$, $\beta(x) > 0$ in (a, b). Then if $\rho(x)$ is any function, positive on (a, b), the right hand side of (6.1) can be written as $\rho(x)\, Df(x)$ with

$$Df = \frac{1}{w_2}\left(\frac{1}{w_1}f'\right)', \qquad \frac{1}{w_1} = \exp q, \qquad \frac{1}{w_2} = \frac{\beta}{\rho}\exp(-q),$$
$$q(x) = \int_c^x \frac{\alpha(t)}{\beta(t)}\, dt, \qquad (6.2)$$

where c is some fixed point in (a, b).

As a first example, we consider the operators of Cheney and Sharma [4],

$$P_n(f, x) = (1 - x)^{n+1} \exp\left(\frac{tx}{1 - x}\right) \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu + 1}\right) L_\nu^{(n)}(t)\, x^\nu, \qquad 0 \leqslant x \leqslant 1, \qquad (6.3)$$

where $t \leqslant 0$ is a parameter, and L_ν is the Laguerre polynomial of degree ν. For $t = 0$, P_n reduce to the operators of Meyer–König and Zeller. For the operators (6.3), Watanabe and Suzuki [23] established relation (6.1) with $\lambda_n = 2(n + 1)$, $\alpha(x) = -2tx$, and $\beta(x) = x(1 - x)^2$. To illustrate the technique we shall give two different saturation results for P_n.

Choosing $\rho(x) = 1$, the saturation class of the Cheney–Sharma operators P_n with respect to the relation

$$2(n + 1)|\, P_n(f, x) - f(x)| \leqslant M + o(1)$$

consists precisely of all f with absolutely continuous f', for which

$$|-2txf'(x) + x(1 - x)^2 f''(x)| \leqslant M, \qquad \text{a.e.} \qquad (6.4)$$

On the other hand, if we choose $\rho(x) = x(1-x)^2 e^{-q(x)}$, where

$$q(x) = \int_{1/2}^{x} \frac{-2t}{(1-s)} ds,$$

then $w_2 \equiv 1$ and $1/w_1 = e^{q(x)}$. We conclude that the saturation class of P_n relative to the relation

$$2(n+1)| P_n(f, x) - f(x)| \leqslant x(1-x)^2 e^{-q(x)} M + o(1) \tag{6.5}$$

consists of all functions f such that $f' e^q$ is absolutely continuous and belongs to the classical Lipschitz class $\mathrm{Lip}_M 1$.

In the special case $t = 0$, the second result above yields a theorem of Watanabe and Suzuki [22] for the Meyer–König and Zeller operators. The saturation results for the operators of Szász, and of Baskakov, proved in [8] and [21] by other methods, also follow from Theorem 4.2.

The Bernstein polynomials $B_n(f, x)$ satisfy relation (6.1) with $\lambda_n = n$, $\alpha = 0$ and $\beta(x) = x(1-x)/2$. The choice $\rho(x) = \frac{1}{2}x(1-x)$ leads to the theorem of Lorentz [13] mentioned in the introduction. Other choices of ρ lead to new saturation theorems. For example, taking $\rho(x) \equiv 1$ and $Df(x) = \frac{1}{2}x(1-x)f''(x)$, we obtain

$$n \mid B_n(f, x) - f(x)| \leqslant M, \qquad 0 < x < 1, \tag{6.6}$$

if and only if f' is absolutely continuous and

$$|f''(x)| \leqslant 2M/[x(1-x)] \qquad \text{a.e.}$$

Still another example of a sequence of positive linear operators satisfying an asymptotic relation of the form (6.1) with $\alpha \neq 0$ can be found in Mühlbach [16].

For functions $f(x, y)$ of two variables, there exist natural definitions of their Bernstein polynomials $B_n(f; x, y)$ for the case of the square, $0 \leqslant x, y \leqslant 1$, and of the triangle; $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 1$. Compare [12, p. 51, formulas (12) and (13), respectively]. Stancu [20, 19] has obtained for the two cases the relation

$$\lim_{n\to\infty} n[B_n(f; x, y) - f(x, y)] = Df(x, y), \tag{6.7}$$

where

$$Df(x, y) = \frac{x(1-x)}{2} \frac{\partial^2 f}{\partial x^2} + \frac{y(1-y)}{2} \frac{\partial^2 f}{\partial y^2} \tag{6.8}$$

for the square and

$$Df(x, y) = \frac{x(1-x)}{2} \frac{\partial^2 f}{\partial x^2} + xy \frac{\partial^2 f}{\partial x \, \partial y} + \frac{y(1-y)}{2} \frac{\partial^2 f}{\partial y^2} \qquad (6.9)$$

for the triangle. The differential operators (6.8) and (6.9) are elliptic, and results of our Section 5 apply. (We note that in [19] Stancu assumes only the existence and continuity of the derivatives appearing in (6.9), which is not enough, instead of $f \in C^2$).

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